Parametric And Polar Review Math 261

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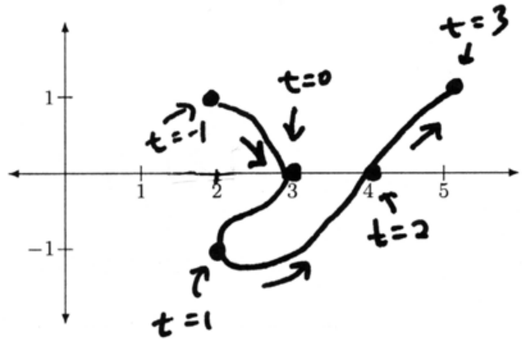
Parametric Equations

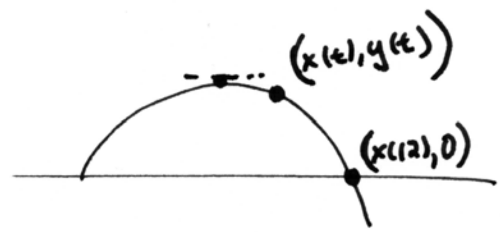
1. Use the table of values to sketch the parametric curve $(x(t), y(t))$, indicating the direction of motion

t	-1	0	1	2	3
x	2	3	2	4	5
y	1	0	-1	0	1



2. A particle follows the trajectory

$$\begin{cases} x(t) = 7t + 9 \\ y(t) = -t^2 + 12t \end{cases}$$



with t in seconds and distance in centimeters.

(a) What is the particles maximum height?

max height when $\frac{dy}{dt} = 0$

$-2t + 12 = 0$

$t = 6$ sec

max height is

$y(6) = -6^2 + 12 \cdot 6$

$= -36 + 72$

$= 36$ cm

(b) When does the particle hit the ground, and how far from the origin does it land?

hits ground when $y(t) = 0$

when $-t^2 + 12t = 0$

$t(-t + 12) = 0$

when $t = 0$ or $t = 12$

start end

hits ground AT $x(12) = 7 \cdot 12 + 9 = 93$ cm

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3. Some modern cell phones are able to track your change in x and y position without using a GPS, by means of their built-in accelerometers and a little bit of Calculus 1.

   Suppose you are walking directly up a hill, and that your position function is given by $c(t) = (7t - 10, -t^3 + 30t^2)$.

   (a) Find the slope of the hill at $t = 5$ during your walk, without eliminating the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3t^2+60t}{7}$$

at $t=5$, $\frac{dy}{dx} = \frac{-3\cdot 5^2+60\cdot 5}{7}$

$$\frac{dy}{dx} = \frac{375}{7}$$

   (b) Eliminate the parameter to find the height of the hill as a function of your horizontal position.

$$\begin{cases} x = 7t - 10 \\ y = -t^3 + 30t^2 \end{cases}$$

$x = 7t - 10$

$\Rightarrow 7\cdot t = x + 10$

$\Rightarrow t = \frac{x+10}{7}$

so

$$y = -\left(\frac{x+10}{7}\right)^3 + 30\left(\frac{x+10}{7}\right)^2$$

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4. Find an equation for the line tangent to the curve $c(t) = \Big(\sin(2t), \cos(4t)\Big)$ at the point when $t = \pi/3$

$$\begin{cases} x = \sin(2t) \\ y = \cos(4t) \end{cases}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\cos(4t)\right]}{\frac{d}{dt}\left[\sin(2t)\right]} = \frac{-\sin(4t)\cdot 4}{\cos(2t)\cdot 2}$$

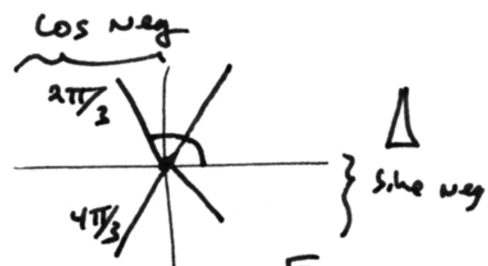
when $t = \pi/3$

$$\left.\frac{dy}{dx}\right|_{\pi/3} = \frac{-\sin(4\pi/3)\cdot 4}{\cos(2\pi/3)\cdot 2}$$

$$= \frac{-\left(-\frac{\sqrt{3}}{2}\right)\cdot 4}{\left(-\frac{1}{2}\right)\cdot 2}$$

$$= \frac{2\sqrt{3}}{-1}$$

$$= -2\sqrt{3}$$



$\sin(\pi/3) = \frac{\sqrt{3}}{2}$

$\cos(\pi/3) = \frac{1}{2}$

$$x(\pi/3) = \sin(2\cdot\pi/3) = +\frac{\sqrt{3}}{2}$$

$$y(\pi/3) = \cos(4\cdot\pi/3) = -\frac{1}{2}$$

$$y = m(x - x_0) + y_0$$

$$y = -2\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)$$

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5. Find the points where the curve $c(t) = (t^2 - 4t, t^3 - 12t)$ has horizontal tangents and vertical tangents.

$$\begin{cases} x = t^2 - 4t \\ y = t^3 - 12t \end{cases}$$

$$\frac{dx}{dt} = 2t - 4 = 2(t-2)$$

$$\frac{dy}{dt} = 3t^2 - 12 = 3(t^2-4) = 3(t+2)(t-2)$$

horizontal tangent

$\Leftrightarrow$

$\frac{dx}{dt} \neq 0$ and $\frac{dy}{dt} = 0$ $\longleftrightarrow$

Note: $\frac{dy}{dt} = 0$ when $t = 2$ or $t = -2$

but

$\frac{dx}{dt} \neq 0$ only for $t = -2$

so only horizontal tangent when $t = -2$

at $((-2)^2 - 4(-2), (-2)^3 - 12\cdot(-2))$

$= (8+8, 8+24)$

$= (16, 32)$

vertical tangent

$\Leftrightarrow$

$\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0$

Note: $\frac{dx}{dt} = 0$ when $t = 2$

but $\frac{dy}{dt} = 0$ when $t = 2$ $\Rightarrow$ no vertical tangents.

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7. Parametrize the following line segment from $(5, 9)$ to $(12, 3)$ using t in $[0, 1]$

Given: $\Delta x = 12-5 = 7$

$\Delta y = 3-9 = -6$

Pick: $\Delta t = 1$

then $a = \frac{\Delta x}{\Delta t} = 7$

$b = \frac{\Delta y}{\Delta t} = -6$

So the parametrization is

$$\begin{cases} x(t) = 7t+5 \\ y(t) = -6t+9 \end{cases}$$

8. Parametrize the following line segment from $(0, 3)$ to $(2, 1)$ using t in $[0, 2]$

Given $\Delta x = 2-0 = 2$

$\Delta y = 1-3 = -2$

Pick $\Delta t = 2$

then $a = \frac{\Delta x}{\Delta t} = \frac{2}{2} = 1$

$b = \frac{\Delta y}{\Delta t} = \frac{-2}{2} = -1$

So the parametrization is

$$\begin{cases} x(t) = 1\cdot t + 0 \\ y(t) = -1\cdot t + 3 \end{cases}$$

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9. Obtain a Cartesian equation for the path by eliminating the parameter

$$\begin{cases} x(\theta) = 1 + 3\sin(\theta) \\ y(\theta) = -2 + 4\cos(\theta) \end{cases}$$

What is the shape of the curve traced out by this path?

Know: $\sin^2\theta + \cos^2\theta = 1$

Know $x = 1 + 3\cdot\sin\theta \Rightarrow \sin\theta = \frac{x-1}{3}$

$y = -2 + 4\cdot\cos\theta \Rightarrow \cos\theta = \frac{y+2}{4}$

So $\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+2}{4}\right)^2 = 1$

this is an ellipse around $(1,-2)$.

10. Find a parametrization of the curve $(x+1)^2 + (y-2)^2 = 25$. What is the shape of this curve?

Know: $(\sin t)^2 + (\boxed{\cos t})^2 = 1$

know: $(x+1)^2 + (y-2)^2 = 25 \quad {}^{,,}5^2$

rewrite $\frac{(x+1)^2}{5^2} + \frac{(y-2)^2}{5^2} = 1$

this is a circle around $(-1,2)$

$\left(\frac{x+1}{5}\right)^2 + \left(\boxed{\frac{y-2}{5}}\right)^2 = 1$

pick $\sin t = \frac{x+1}{5}$

$\boxed{\cos t} = \boxed{\frac{y-2}{5}}$

solving gives

$$\begin{cases} x = 5\cdot\sin t - 1 \\ y = 5\cdot\cos t + 2 \end{cases}$$

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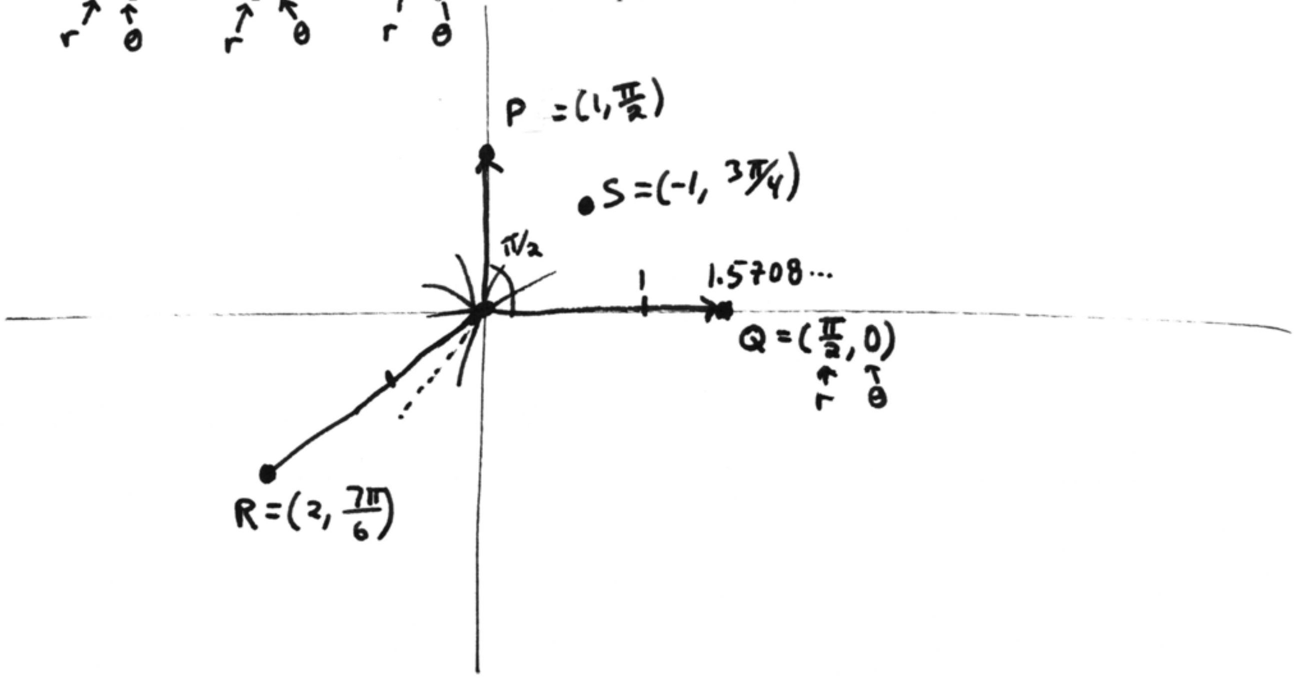
Polar Coordinates, Equations, and Area

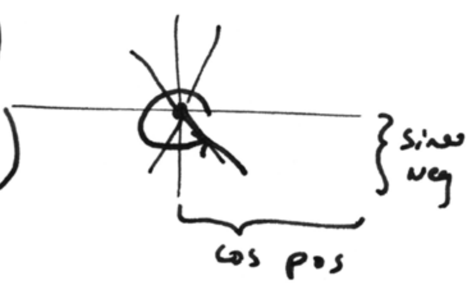
1. Plot and label the points with the following polar coordinates:
$P = (1, \frac{\pi}{2})$, $Q = (\frac{\pi}{2}, 0)$, $R = (2, \frac{7\pi}{6})$, $S = (-1, \frac{3\pi}{4})$.

r θ r θ r θ



2. Find the Cartesian (rectangular) coordinates for the point with polar coordinate
$(r,\theta) =$ $(3, \frac{5\pi}{3})$

$$\begin{cases} x = r\cdot\cos\theta = 3\cdot\cos\left(\frac{5\pi}{3}\right) = 3\cdot\left(+\frac{1}{2}\right) \\ y = r\cdot\sin\theta = 3\cdot\sin\left(\frac{5\pi}{3}\right) = 3\cdot\left(-\frac{\sqrt{3}}{2}\right) \end{cases}$$



$x = \frac{3}{2}$

$y = \frac{-3\sqrt{3}}{2}$

AKA: $\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$

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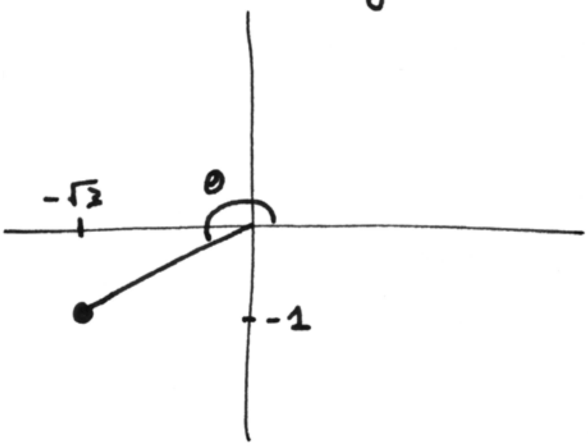
3. Find a polar coordinate representation of the point with Cartesian (rectangular) coordinates $(x, y) = (-\sqrt{3}, -1)$.



$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

Know

$\tan\theta = \frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$

Know

$\tan(\pi/6) = \frac{1}{\sqrt{3}}$

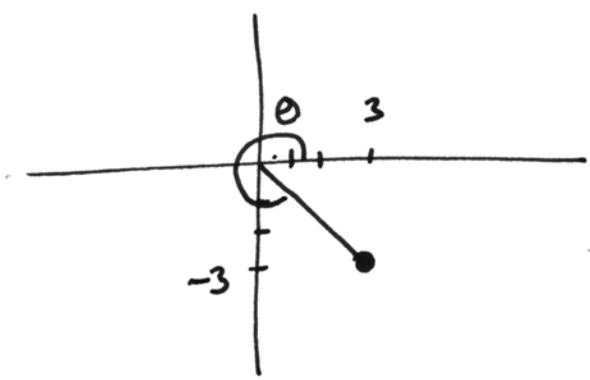
$\Rightarrow$ inside $\measuredangle$ is $\pi/6$

Choose

$\theta = 7\pi/6$

$\boxed{\text{Polar coordinates are } (r, \theta) = \left(2, \tfrac{7\pi}{6}\right)}$

4. Find a polar coordinate representation of the point with Cartesian (rectangular) coordinates $(x, y) = (3, -3)$.



$r = \sqrt{3^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18}$

Know: $\tan\theta = \frac{y}{x} = \frac{-3}{3} = -1$

Know: $\tan(\pi/4) = 1$

$\Rightarrow$ inside $\measuredangle$ is $\pi/4$

Choose $\theta = \frac{7\pi}{4}$

$\boxed{\text{Polar coordinates are } (r, \theta) = \left(\sqrt{18}, \tfrac{7\pi}{4}\right)}$